

REGION OF VARIABILITY FOR CERTAIN CLASSES OF UNIVALENT FUNCTIONS SATISFYING DIFFERENTIAL INEQUALITIES

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ABSTRACT. For complex numbers α, β and $M \in \mathbb{R}$ with $0 < M \leq |\alpha|$ and $|\beta| \leq 1$, let $\mathcal{B}(\alpha, \beta, M)$ be the class of analytic and univalent functions f in the unit disk \mathbb{D} with $f(0) = 0$, $f'(0) = \alpha$ and $f''(0) = M\beta$ satisfying $|zf''(z)| \leq M$, $z \in \mathbb{D}$. Let $\mathcal{P}(\alpha, M)$ be the another class of analytic and univalent functions in \mathbb{D} with $f(0) = 0$, $f'(0) = \alpha$ satisfying $\operatorname{Re}(zf''(z)) > -M$, $z \in \mathbb{D}$, where $\alpha \in \mathbb{C} \setminus \{0\}$, $0 < M \leq 1/\log 4$. For any fixed $z_0 \in \mathbb{D}$ and $\lambda \in \overline{\mathbb{D}}$ we shall determine the region of variability V_j ($j = 1, 2$) for $f'(z_0)$ when f ranges over the class \mathcal{S}_j ($j = 1, 2$), where

$$\mathcal{S}_1 = \left\{ f \in \mathcal{B}(\alpha, \beta, M) : f'''(0) = M(1 - |\beta|^2)\lambda \right\}$$

and

$$\mathcal{S}_2 = \{f \in \mathcal{P}(\alpha, M) : f''(0) = 2M\lambda\}.$$

In the final section we graphically illustrate the region of variability for several sets of parameters.

1. INTRODUCTION AND PRELIMINARIES

We denote the class of all analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by $\mathcal{H}(\mathbb{D})$, and think of $\mathcal{H}(\mathbb{D})$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of \mathbb{D} . We begin with the discussion of some properties of families of analytic functions considered as subsets of $\mathcal{H}(\mathbb{D})$. A univalent function f is called starlike if $f(\mathbb{D})$ is a starlike domain (w.r.t. origin). Let \mathcal{S}^* denote the class of starlike functions $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = 0 = f'(0) - 1$. Denote by \mathcal{K} the subclass of functions $f \in \mathcal{H}(\mathbb{D})$ with $f(0) = 0 = f'(0) - 1$ such that f maps \mathbb{D} conformally onto a convex domain. If

$$\mathcal{B}(M) = \{f \in \mathcal{H}(\mathbb{D}) : f(0) = f'(0) - 1 = 0 \text{ and } |zf''(z)| \leq M\},$$

then it is known that $\mathcal{B}(M) \subsetneq \mathcal{K}$ if $0 < M \leq 1/2$, and $\mathcal{B}(M) \subsetneq \mathcal{S}^*$ if $0 < M \leq 1$ and the inclusions are sharp. For a general result we refer to [5]. In this paper, we

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are interested in two subclasses of analytic functions and use the Schwarz lemma as the main tool in describing the boundary behavior of these two classes of functions.

1.1. The Class $\mathcal{B}(\alpha, \beta, M)$. Let $\alpha, \beta \in \mathbb{C}$ and $M \in \mathbb{R}$ such that $0 < M \leq |\alpha|$ and $|\beta| \leq 1$. Let $\mathcal{B}(\alpha, \beta, M)$ denote the class of all functions f analytic and univalent in the unit disk \mathbb{D} , with $f(0) = 0$, $f'(0) = \alpha$, and $f''(0) = M\beta$ satisfying

$$(1.2) \quad |zf''(z)| \leq M, \quad z \in \mathbb{D}.$$

Note that $\alpha \neq 0$. If $f \in \mathcal{B}(\alpha, \beta, M)$, then we may write

$$zf''(z) = M\omega(z)$$

for some $\omega \in \mathcal{B}_0$, where \mathcal{B}_0 denotes the class of functions ω analytic in \mathbb{D} such that $|\omega(z)| \leq 1$ in \mathbb{D} and $\omega(0) = 0$. This gives the representation

$$f'(z) - \alpha = f'(z) - f'(0) = M \int_0^1 \frac{\omega(tz)}{t} dt$$

so that, by integration,

$$f(z) = \alpha z + Mz \int_0^1 \frac{(1-t)\omega(tz)}{t} dt.$$

By the Schwarz lemma, we have $|\omega(z)| \leq |z|$ and so that previous relation gives that

$$|f'(z) - \alpha| \leq M|z| < M, \quad z \in \mathbb{D}$$

which, in particular, shows that functions in $\mathcal{B}(\alpha, \beta, M)$ are univalent in \mathbb{D} if $M \leq |\alpha|$. It is easy to see that functions in $\mathcal{B}(\alpha, \beta, M)$ are not necessarily univalent if $M > |\alpha|$. This fact may be demonstrated by, for example, the function

$$f(z) = \alpha z + (M/2)z^2.$$

Furthermore, every $f \in \mathcal{B}(\alpha, \beta, M)$ can be associated with a function ω_f in \mathcal{B}_0 and this association is clearly given by

$$(1.3) \quad \omega_f(z) = \frac{f''(z) - M\beta}{M - \overline{\beta}f''(z)}, \quad z \in \mathbb{D}.$$

A simple application of the Schwarz lemma shows that if $f \in \mathcal{B}(\alpha, \beta, M)$ then one has $|\omega'_f(0)| \leq 1$ which, in particular, gives a restriction on $f'''(0)$. Indeed, it is a simple exercise to see that

$$(1.4) \quad f'''(0) = M(1 - |\beta|^2)\omega'_f(0)$$

and therefore, with $\omega'_f(0) = \lambda$, we have $|f'''(0)| \leq M(1 - |\beta|^2)$. Using (1.3) and (1.4), one can obtain by a computation that

$$(1.5) \quad M(1 - |\beta|^2)\omega''_f(0) = 2M(1 - |\beta|^2)\lambda^2\overline{\beta} + (1 + \lambda\overline{\beta})f^{(iv)}(0).$$

Also if we let

$$(1.6) \quad g(z) = \frac{\frac{\omega_f(z)}{z} - \lambda}{1 - \overline{\lambda}\frac{\omega_f(z)}{z}}, \quad \text{for } |\lambda| < 1,$$

and $g(z) = 0$ for $|\lambda| = 1$, then $g \in \mathcal{B}_0$, and we compute that

$$g'(0) = \begin{cases} \frac{1}{1-|\lambda|^2} \left(\frac{\omega_f(z)}{z} \right)' \Big|_{z=0} = \frac{1}{1-|\lambda|^2} \left(\frac{\omega_f''(0)}{2} \right) & \text{for } |\lambda| < 1 \\ 0 & \text{for } |\lambda| = 1. \end{cases}$$

For convenience, we set $g'(0) = a$. From (1.5) we note that for $|\lambda| < 1$,

$$(1.7) \quad |g'(0)| \leq 1 \iff f^{(iv)}(0) = \frac{2M(1-|\beta|^2)}{1+\lambda\bar{\beta}} [(1-|\lambda|^2)a - \lambda^2\bar{\beta}]$$

for some $a \in \overline{\mathbb{D}}$. Observe that $a = 0$ when $|\lambda| = 1$, and $|a| < 1$ if and only if $|\lambda| < 1$, according to the Schwarz lemma.

1.8. The Class $\mathcal{P}(\alpha, M)$. Another class of analytic functions of our interest is defined by

$$(1.9) \quad \mathcal{P}(\alpha, M) = \{f \in \mathcal{H}(\mathbb{D}) : f(0) = 0, f'(0) = \alpha \text{ and } \operatorname{Re} z f''(z) > -M, z \in \mathbb{D}\}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, and $0 < M \leq 1/\log 4$. In [1], it has been shown that

$$\mathcal{P}(1, M) \subset \mathcal{S}^* \quad \text{for } 0 < M \leq \frac{1}{\log 4}.$$

For any larger value of M , functions in $\mathcal{P}(1, M)$ are not necessarily locally univalent. Later in [2, Theorem 2.10] the authors have proved that

$$\mathcal{P}(1, M) \subset \mathcal{S}^*(\alpha)$$

($0 \leq \alpha \leq \frac{1}{2}$) whenever

$$0 \leq M \leq \frac{1-2\alpha}{2\alpha + \log 4}.$$

This generalizes the last relation. However, the Herglotz representation for analytic functions with positive real part in \mathbb{D} shows that if $f \in \mathcal{P}(\alpha, M)$, then there exists a unique positive unit measure μ on $(-\pi, \pi]$ such that

$$\frac{zf''(z) + M}{M} = \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t), \quad \text{i.e.} \quad f''(z) = 2M \int_{-\pi}^{\pi} \frac{e^{-it}}{1 - ze^{-it}} d\mu(t).$$

Integrating from 0 to z shows that

$$f'(z) = 2M \int_{-\pi}^{\pi} \log \left(\frac{1}{1 - ze^{-it}} \right) d\mu(t) + \alpha.$$

Once again integrating the above from 0 to z gives the following representation

$$f(z) = 2M \int_{-\pi}^{\pi} \{z - (z - e^{it}) \log(1 - ze^{-it})\} d\mu(t) + \alpha z.$$

Functions of the above form belong to the class $\mathcal{P}(\alpha, M)$. Clearly, for every $f \in \mathcal{P}(\alpha, M)$, there exists an $\omega_f \in \mathcal{B}_0$ such that

$$(1.10) \quad \omega_f(z) = \frac{zf''(z)}{zf''(z) + 2M}, \quad z \in \mathbb{D}.$$

Note that $|\omega'_f(0)| \leq 1$. By the Schwarz lemma it is a simple exercise to see that if $f \in \mathcal{P}(\alpha, M)$, then

$$(1.11) \quad f''(0) = 2M\omega'_f(0)$$

and therefore, with $\omega'_f(0) = \lambda$, we have $|f''(0)| \leq 2M$. Using (1.10), one can obtain by a computation that

$$(1.12) \quad \omega''_f(0) = \frac{f'''(0)}{M} - 2\lambda^2.$$

Thus, if g is defined by (1.6) with $\omega_f(z)$ as in (1.10), then it follows that

$$|g'(0)| \leq 1 \iff f'''(0) = 2M((1 - |\lambda|^2)a + \lambda^2)$$

for some $a \in \overline{\mathbb{D}}$. Again we remark that $|g'(0)| = 0$ occurs if and only if $a = \lambda$ with $|\lambda| = 1$. Also, $|a| < 1$ if and only if $|\lambda| < 1$, according to the Schwarz lemma.

For $\lambda \in \overline{\mathbb{D}}$ and for each fixed $z_0 \in \mathbb{D}$, we introduce the following sets:

$$\begin{aligned} \mathcal{S}_1(\lambda) &= \left\{ f \in \mathcal{B}(\alpha, \beta, M) : f'''(0) = M(1 - |\beta|^2)\lambda \right\}, \\ \mathcal{S}_2(\lambda) &= \left\{ f \in \mathcal{P}(\alpha, M) : f''(0) = 2M\lambda \right\}, \end{aligned}$$

and

$$V_j(z_0, \lambda) = \{f'(z_0) : f \in \mathcal{S}_j(\lambda)\} \quad \text{for } j = 1, 2.$$

The purpose of the present paper is to determine explicitly the region of variability $V_j(z_0, \lambda)$ of $f'(z_0)$ when f ranges over the class $\mathcal{S}_j(\lambda)$ ($j = 1, 2$). Questions of this nature have been discussed recently in [6, 7, 8, 12].

2. THE BASIC PROPERTIES OF $V_1(z_0, \lambda)$, $V_2(z_0, \lambda)$ AND THE MAIN RESULTS

For a positive integer p , let

$$(\mathcal{S}^*)^p = \{f = f_0^p : f_0 \in \mathcal{S}^*\}$$

and recall the following well-known result whose analytic proof is given in [11] (see also [3, 4]).

Lemma 2.1. *Let f be an analytic function in \mathbb{D} with $f(z) = z^p + \dots$. If*

$$\operatorname{Re} \left(1 + z \frac{f''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D},$$

then $f \in (\mathcal{S}^)^p$.*

For the sake of convenience, we use the notation $V_j(z_0, \lambda) = V_j$ and $\mathcal{S}_j(\lambda) = \mathcal{S}_j$ for $j = 1, 2$. Now, we begin our investigation by stating certain general properties of the set V_j ($j = 1, 2$).

Proposition 2.2. *We have*

- (1) V_1 is compact
- (2) V_1 is convex

(3) for $|\lambda| = 1$ or $z_0 = 0$,

$$(2.3) \quad V_1 = \left\{ \frac{M}{\beta} \left(z_0 - (1 - |\beta|^2) \frac{\log(1 + \lambda \bar{\beta} z_0)}{\lambda \bar{\beta}} \right) + \alpha \right\} \quad \text{if } \beta \neq 0;$$

and for $\beta = 0$, V_1 is obtained as a limiting case from (2.3)

(4) for $|\lambda| < 1$ and $z_0 \in \mathbb{D} \setminus \{0\}$, V_1 has

$$\frac{M}{\beta} \left(z_0 - (1 - |\beta|^2) \frac{\log(1 + \lambda \bar{\beta} z_0)}{\lambda \bar{\beta}} \right) + \alpha$$

as an interior point.

Proof. (1) First we show that \mathcal{S}_1 is compact. For this, we need to prove that for $\{f_n\}$ in \mathcal{S}_1 , whenever $f_n \rightarrow f$ uniformly on every compact subset of \mathbb{D} , $f \in \mathcal{S}_1$. We recall that if $f_n \rightarrow f$ uniformly on every compact subset of \mathbb{D} then $f'_n \rightarrow f'$ uniformly on every compact subset of \mathbb{D} . Thus, if $f_n \rightarrow f$ uniformly, $f_n(0) \rightarrow f(0)$ pointwise which gives $f(0) = 0$. Repeated use of this fact for derivatives, we conclude that $f'_n(0) \rightarrow f'(0)$ pointwise and therefore, $f'(0) = \alpha$. Similarly, $f''(0) = M\beta$, $f'''(0) = M(1 - |\beta|^2)\lambda$ and $f^{(iv)}(0)$ satisfies

$$f^{(iv)}(0) = \frac{2M(1 - |\beta|^2)}{1 + \lambda \bar{\beta}} [(1 - |\lambda|^2)a - \lambda^2 \bar{\beta}].$$

Also, we have

$$|zf''_n(z)| \rightarrow |zf''(z)|$$

uniformly on every compact subset of \mathbb{D} . Since each member of the sequence $\{f_n\}$ is in \mathcal{S}_1 , it follows that $|zf''(z)| \leq M$. We conclude that \mathcal{S}_1 is compact.

Finally, for fixed $z_0 \in \mathbb{D}$, define $\psi : \mathcal{S}_1 \rightarrow V_1$ by

$$\psi(f)(z_0) = f'(z_0).$$

Clearly ψ is continuous. Thus, V_1 is compact.

(2) If f_0 and f_1 belong to \mathcal{S}_1 , then, for $0 \leq t \leq 1$, the function f_t defined by

$$f_t(z) = \int_0^z \{(1-t)f'_0(\zeta) + tf'_1(\zeta)\} d\zeta$$

also belongs to \mathcal{S}_1 . Clearly, we have

$$f'_t(z) = (1-t)f'_0(z) + tf'_1(z),$$

and the convexity of V_1 is evident.

(3) If $z_0 = 0$, (2.3) trivially holds. If $|\lambda| = 1$, then from (1.4) we see that $|\omega'_f(0)| = 1$ and therefore it follows from the Schwarz lemma that $\omega_f(z) = \lambda z$, which by (1.3) gives that

$$\frac{zf''(z)}{M} = \frac{(\lambda z + \beta)z}{1 + \bar{\beta}\lambda z}, \quad \text{for } |\beta| \leq 1.$$

By integrating the above from 0 to z_0 we see that

$$\begin{aligned} f'(z_0) &= M \int_0^{z_0} \frac{\lambda\zeta + \beta}{1 + \overline{\beta}\lambda\zeta} d\zeta + \alpha \quad \text{if } \beta \neq 0 \\ &= \frac{M}{\beta} \int_0^{z_0} \left[\left(1 - \frac{1}{1 + \overline{\beta}\lambda\zeta}\right) + \frac{\beta}{\lambda} \left(\frac{\lambda\overline{\beta}}{1 + \overline{\beta}\lambda\zeta}\right) \right] d\zeta + \alpha \quad \text{if } \beta \neq 0 \end{aligned}$$

and a computation gives

$$f'(z_0) = \begin{cases} \frac{Mz_0}{\beta} - \frac{M}{\lambda\overline{\beta}^2}(1 - |\beta|^2) \log(1 + \lambda\overline{\beta}z_0) + \alpha & \text{if } \beta \neq 0 \\ \alpha + \frac{M\lambda}{2}z_0^2 & \text{if } \beta = 0 \end{cases}$$

and thus,

$$V_1 = \begin{cases} \frac{Mz_0}{\beta} - \frac{M}{\lambda\overline{\beta}^2}(1 - |\beta|^2) \log(1 + \lambda\overline{\beta}z_0) + \alpha & \text{if } \beta \neq 0. \end{cases}$$

We remark that

$$\lim_{\beta \rightarrow 0} \left\{ \frac{Mz_0}{\beta} - \frac{M}{\lambda\overline{\beta}^2}(1 - |\beta|^2) \log(1 + \lambda\overline{\beta}z_0) + \alpha \right\} = \alpha + \frac{M\lambda}{2}z_0^2$$

and, therefore for $\beta = 0$,

$$V_1 = \left\{ \alpha + \frac{M\lambda}{2}z_0^2 \right\}.$$

Hence, the extremal function in \mathcal{S}_1 for $|\lambda| = 1$ is of the form

$$f(z) = \begin{cases} \frac{M}{\beta} \left\{ \frac{z^2}{2} - \frac{(1 - |\beta|^2)}{\lambda\overline{\beta}} \left[\left(z + \frac{1}{\lambda\overline{\beta}} \right) \log(1 + \lambda\overline{\beta}z) - z \right] \right\} + \alpha z & \text{if } 0 < |\beta| \leq 1 \\ \alpha z + \frac{M\lambda}{6}z^3 & \text{if } \beta = 0. \end{cases}$$

(4) For $\lambda \in \mathbb{D}$, we let

$$(2.4) \quad \delta(z, \lambda) = \frac{z + \lambda}{1 + \overline{\lambda}z}.$$

A simplification of (1.6) with $g(z) = az$ ($|a| < 1$) leads to

$$(2.5) \quad F_{a,\lambda}(z) := f(z) = \int_0^z \left\{ \int_0^{\zeta_2} \left(\frac{M(\delta(a\zeta_1, \lambda)\zeta_1 + \beta)}{1 + \overline{\beta}\delta(a\zeta_1, \lambda)\zeta_1} \right) d\zeta_1 \right\} d\zeta_2 + \alpha z,$$

where $z \in \mathbb{D}$.

First we claim that $F_{a,\lambda} \in \mathcal{S}_1$. In fact, with the aid of (2.5) we easily get

$$zF_{a,\lambda}''(z) = \frac{M(\delta(az, \lambda)z + \beta)z}{1 + \overline{\beta}\delta(az, \lambda)z}.$$

As $\delta(az, \lambda)$ lies in the unit disk \mathbb{D} , $F_{a,\lambda} \in \mathcal{S}_1$. Further, one may verify that

$$(2.6) \quad \omega_{F_{a,\lambda}}(z) = z\delta(az, \lambda).$$

Next our claim is that for a fixed $\lambda \in \mathbb{D}$ and $z_0 \in \mathbb{D} \setminus \{0\}$,

$$\mathbb{D} \ni a \mapsto F'_{a,\lambda}(z_0) = \int_0^{z_0} \left(\frac{M(\delta(a\zeta, \lambda)\zeta + \beta)}{1 + \bar{\beta}\delta(a\zeta, \lambda)\zeta} \right) d\zeta + \alpha,$$

is a non-constant analytic function of $a \in \mathbb{D}$, and hence is an open mapping.

Finally, we claim that the mapping $\mathbb{D} \ni a \mapsto F'_{a,\lambda}(z_0)$ is a non-constant analytic function of a for each fixed $z_0 \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \mathbb{D}$. For this, we put

$$h_1(z) = \frac{3}{M(1 - |\beta|^2)(1 - |\lambda|^2)} \frac{\partial}{\partial a} \left\{ F'_{a,\lambda}(z) \right\} \Big|_{a=0}$$

and obtain that

$$h_1(z) = 3 \int_0^z \frac{\zeta^2}{(1 + \bar{\beta}\lambda\zeta)^2} d\zeta = z^3 + \dots$$

which gives

$$\operatorname{Re} \left\{ \frac{zh_1''(z)}{h_1'(z)} \right\} = 2 \operatorname{Re} \left\{ \frac{1}{1 + \bar{\beta}\lambda z} \right\} \geq \frac{2}{1 + |\lambda||\beta|} > 1, \quad z \in \mathbb{D}.$$

By Lemma 2.1, there exists a function $h_0 \in \mathcal{S}^*$ with $h_1 = h_0^3$. The univalence of h_0 and $h_0(0) = 0$ imply that $h_1(z_0) \neq 0$ for $z_0 \in \mathbb{D} \setminus \{0\}$. Consequently, the mapping $\mathbb{D} \ni a \mapsto F'_{a,\lambda}(z_0)$ is a non-constant analytic function of a . Thus

$$F'_{0,\lambda}(z_0) = \int_0^{z_0} M \left(\frac{\lambda\zeta + \beta}{1 + \bar{\beta}\lambda\zeta} \right) d\zeta + \alpha$$

is an interior point of $\{F'_{a,\lambda}(z_0) : a \in \mathbb{D}\} \subset V_1$. □

We have the following analog of Proposition 2.2 for functions in $\mathcal{P}(\alpha, M)$.

Proposition 2.7. *We have*

- (1) V_2 is compact
- (2) V_2 is convex
- (3) for $|\lambda| = 1$ or $z_0 = 0$,

$$(2.8) \quad V_2 = \{-2M \log(1 - \lambda z_0) + \alpha\}$$

- (4) for $|\lambda| < 1$ and $z_0 \in \mathbb{D} \setminus \{0\}$, V_2 has $-2M \log(1 - \lambda z_0) + \alpha$ as an interior point

Proof. Part (1) and (2) follows exactly as in the proof of Proposition 2.2 and so we omit the details.

(3) If $z_0 = 0$, (2.8) trivially holds. If $|\lambda| = 1$, then for $f \in \mathcal{P}(\alpha, M)$ it follows that $f''(0) = 2M\omega_f'(0)$ and from (1.11) we see that $|\omega_f'(0)| = 1$ and therefore it follows from the Schwarz lemma that $\omega_f(z) = \lambda z$, which by (1.10) gives

$$zf''(z) + M = \frac{M(1 + \lambda z)}{1 - \lambda z} \quad \text{or} \quad f''(z) = \frac{2M\lambda}{1 - \lambda z}.$$

By integrating from 0 to z_0 we see that

$$f'(z_0) = -2M \log(1 - \lambda z_0) + \alpha.$$

Thus

$$V_2 = \{-2M \log(1 - \lambda z_0) + \alpha\}.$$

Hence, the extremal function in \mathcal{S}_2 for $|\lambda| = 1$ is of the form

$$f(z) = 2M \left(\frac{1}{\lambda} - z \right) \log \left(\frac{1}{1 - \lambda z} \right) + (\alpha - 2M)z.$$

(4) Let $\lambda \in \mathbb{D}$ and $a \in \mathbb{D}$. A simple computation as before helps to introduce

$$(2.9) \quad H_{a,\lambda}(z) := f(z) = \int_0^z \left\{ \int_0^{\zeta_2} \frac{2M\delta(a\zeta_1, \lambda)}{1 - \delta(a\zeta_1, \lambda)\zeta_1} d\zeta_1 \right\} d\zeta_2 + \alpha z, \quad z \in \mathbb{D},$$

where $\delta(z, \lambda)$ is defined by (2.4). From this we see that $H_{a,\lambda} \in \mathcal{S}_2$ and

$$(2.10) \quad \omega_{H_{a,\lambda}}(z) = z\delta(az, \lambda).$$

For a fixed $\lambda \in \mathbb{D}$ and $z_0 \in \mathbb{D} \setminus \{0\}$, the function

$$\mathbb{D} \ni a \mapsto H'_{a,\lambda}(z_0) = \int_0^{z_0} \frac{2M\delta(a\zeta, \lambda)}{1 - \delta(a\zeta, \lambda)\zeta} d\zeta + \alpha$$

is a non-constant analytic function of $a \in \mathbb{D}$, and hence is an open mapping. We claim that the mapping $\mathbb{D} \ni a \mapsto H'_{a,\lambda}(z_0)$ is a non-constant analytic function of a for each fixed $z_0 \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \mathbb{D}$. For this we let

$$h_2(z) = \frac{1}{M(1 - |\lambda|^2)} \frac{\partial}{\partial a} \left\{ H'_{a,\lambda}(z) \right\} \Big|_{a=0}$$

so that

$$h_2(z) = 2 \int_0^z \frac{\zeta}{(1 - \lambda\zeta)^2} d\zeta = z^2 + \dots.$$

This gives

$$\operatorname{Re} \left\{ \frac{zh_2''(z)}{h_2'(z)} \right\} = \operatorname{Re} \left\{ \frac{1 + \lambda z}{1 - \lambda z} \right\} > 0, \quad z \in \mathbb{D}.$$

By Lemma 2.1 there exists a function $h_0 \in \mathcal{S}^*$ with $h_2 = h_0^2$ so that the univalence of h_0 and $h_0(0) = 0$ imply that $h_2(z_0) \neq 0$ for $z_0 \in \mathbb{D} \setminus \{0\}$. Consequently, the mapping $\mathbb{D} \ni a \mapsto H'_{a,\lambda}(z_0)$ is a non-constant analytic function of a . Thus,

$$H'_{0,\lambda}(z_0) = -2M \log(1 - \lambda z_0) + \alpha$$

is an interior point of $\{H'_{a,\lambda}(z_0) : a \in \mathbb{D}\} \subset V_2$. □

For each $j = 1, 2$, V_j is a compact convex subset of \mathbb{C} and has nonempty interior and therefore, the boundary ∂V_j is a Jordan curve and V_j is the union of ∂V_j and its inner domain. We now state our main results.

Theorem 2.11. For $\lambda \in \mathbb{D}$, the boundary ∂V_1 is the Jordan curve given by

$$(-\pi, \pi] \ni \theta \mapsto F'_{e^{i\theta}, \lambda}(z_0) = \int_0^{z_0} \left(\frac{M(\delta(e^{i\theta}\zeta, \lambda)\zeta + \beta)}{1 + \bar{\beta}\delta(e^{i\theta}\zeta, \lambda)\zeta} \right) d\zeta + \alpha.$$

Here $\alpha, \beta \in \mathbb{C}$ and $M \in \mathbb{R}$ with $0 < M \leq |\alpha|$ and $|\beta| \leq 1$. If $f'(z_0) = F'_{e^{i\theta}, \lambda}(z_0)$ for some $f \in \mathcal{S}_1$, $z_0 \in \mathbb{D} \setminus \{0\}$, and $\theta \in (-\pi, \pi]$, then $f(z) = F_{e^{i\theta}, \lambda}(z)$.

Theorem 2.12. For $\lambda \in \mathbb{D}$, the boundary ∂V_2 is the Jordan curve given by

$$(-\pi, \pi] \ni \theta \mapsto H'_{e^{i\theta}, \lambda}(z_0) = \int_0^{z_0} \frac{2M\delta(e^{i\theta}\zeta, \lambda)}{1 - \delta(e^{i\theta}\zeta, \lambda)\zeta} d\zeta + \alpha$$

where $\alpha \in \mathbb{C} \setminus \{0\}$ and $0 < M \leq 1/\log 4$. If $f'(z_0) = H'_{e^{i\theta}, \lambda}(z_0)$ for some $f \in \mathcal{S}_2$, $z_0 \in \mathbb{D} \setminus \{0\}$ and $\theta \in (-\pi, \pi]$, then $f(z) = H_{e^{i\theta}, \lambda}(z)$.

3. PREPARATION FOR THE PROOF OF THEOREM 2.11

We exclude the case $|\lambda| = 1$ from the following result as this follows from our earlier discussion.

Proposition 3.1. For $f \in \mathcal{S}_1(\lambda)$ with $|\lambda| < 1$, we have

$$(3.2) \quad |f''(z) - c_1(z, \lambda)| \leq r_1(z, \lambda), \quad z \in \mathbb{D},$$

where

$$\begin{aligned} c_1(z, \lambda) &= \frac{M(1 - |z|^2) \{ \beta(1 + |z|^2) + \beta^2 \bar{\lambda} \bar{z} + \lambda z \}}{(1 - |\beta|^2 |z|^4) - (1 - |\beta|^2) |\lambda|^2 |z|^2 + 2(1 - |z|^2) \operatorname{Re}(\bar{\beta} \lambda z)}, \quad \text{and} \\ r_1(z, \lambda) &= \frac{(1 - |\lambda|^2)(1 - |\beta|^2) |z|^2}{(1 - |\beta|^2 |z|^4) - (1 - |\beta|^2) |\lambda|^2 |z|^2 + 2(1 - |z|^2) \operatorname{Re}(\bar{\beta} \lambda z)}. \end{aligned}$$

For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = F_{e^{i\theta}, \lambda}$ for some $\theta \in \mathbb{R}$.

Proof. Let $f \in \mathcal{S}_1$. Then (1.3) holds with $\omega_f \in \mathcal{B}_0$ and $\omega'_f(0) = \lambda$. It follows from the Schwarz lemma that

$$(3.3) \quad \left| \frac{\frac{\omega_f(z)}{z} - \lambda}{1 - \bar{\lambda} \frac{\omega_f(z)}{z}} \right| \leq |z|.$$

From (1.2) and (1.3) this is easily seen to be equivalent to

$$(3.4) \quad \left| \frac{f''(z) - A_1(z, \lambda)}{f''(z) + B_1(z, \lambda)} \right| \leq |z| |\tau_1(z, \lambda)|,$$

where

$$(3.5) \quad \begin{cases} A_1(z, \lambda) = \frac{M(\beta + \lambda z)}{1 + \bar{\beta} \lambda z}, \\ B_1(z, \lambda) = -\frac{M(z + \bar{\lambda} \beta)}{\bar{\beta} z + \bar{\lambda}}, \\ \tau_1(z, \lambda) = \frac{\bar{\beta} z + \bar{\lambda}}{1 + \bar{\beta} \lambda z}. \end{cases}$$

Further, a computation shows that the inequality (3.4) is equivalent to

$$(3.6) \quad \left| f''(z) - \frac{A_1(z, \lambda) + |z|^2 |\tau_1(z, \lambda)|^2 B_1(z, \lambda)}{1 - |z|^2 |\tau_1(z, \lambda)|^2} \right| \leq \frac{|z| |\tau_1(z, \lambda)| |A_1(z, \lambda) + B_1(z, \lambda)|}{1 - |z|^2 |\tau_1(z, \lambda)|^2}.$$

Now we have

$$\begin{aligned} 1 - |z|^2 |\tau_1(z, \lambda)|^2 &= 1 - |z|^2 \left| \frac{\bar{\beta}z + \bar{\lambda}}{1 + \bar{\beta}\lambda z} \right|^2 \\ &= \frac{(1 - |\beta|^2 |z|^4) - (1 - |\beta|^2) |\lambda|^2 |z|^2 + 2(1 - |z|^2) \operatorname{Re}(\bar{\beta}\lambda z)}{|1 + \bar{\beta}\lambda z|^2}, \end{aligned}$$

$$\begin{aligned} A_1(z, \lambda) + B_1(z, \lambda) &= \frac{M(\beta + \lambda z)}{1 + \bar{\beta}\lambda z} - \frac{M(z + \bar{\lambda}\beta)}{\bar{\beta}z + \bar{\lambda}} \\ &= \frac{M(1 - |\lambda|^2)(1 - |\beta|^2)z}{(1 + \bar{\beta}\lambda z)(\bar{\beta}z + \bar{\lambda})} \end{aligned}$$

and

$$\begin{aligned} A_1(z, \lambda) + |z|^2 |\tau_1(z, \lambda)|^2 B_1(z, \lambda) &= \frac{M(\beta + \lambda z)}{1 + \bar{\beta}\lambda z} - |z|^2 \left| \frac{\bar{\beta}z + \bar{\lambda}}{1 + \bar{\beta}\lambda z} \right|^2 \left(\frac{M(z + \bar{\lambda}\beta)}{\bar{\beta}z + \bar{\lambda}} \right) \\ &= \frac{M(1 - |z|^2) \{ \beta(1 + |z|^2) + \beta^2 \bar{\lambda} \bar{z} + \lambda z \}}{|1 + \bar{\beta}\lambda z|^2}. \end{aligned}$$

An easy calculation yields that

$$\frac{A_1(z, \lambda) + |z|^2 |\tau_1(z, \lambda)|^2 B_1(z, \lambda)}{1 - |z|^2 |\tau_1(z, \lambda)|^2} = c_1(z, \lambda)$$

and

$$\frac{|z| |\tau_1(z, \lambda)| |A_1(z, \lambda) + B_1(z, \lambda)|}{1 - |z|^2 |\tau_1(z, \lambda)|^2} = r_1(z, \lambda).$$

Now the inequality (3.2) follows from these equalities and (3.6).

It is easy to see that the equality occurs for $z \in \mathbb{D}$ in (3.2) if and only if equality occurs in (3.3). Thus the equality in (3.2) holds whenever $f = F_{e^{i\theta}, \lambda}$ for some $\theta \in \mathbb{R}$. Conversely if the equality occurs for some $z \in \mathbb{D} \setminus \{0\}$ in (3.2), then the equality must hold in (3.6) and hence (3.3) holds. Thus, from the Schwarz lemma, there exists a $\theta \in \mathbb{R}$ such that $\omega_f(z) = z\delta(e^{i\theta}z, \lambda)$ for all $z \in \mathbb{D}$. This implies $f = F_{e^{i\theta}, \lambda}$. \square

The case $\lambda = 0$ of Proposition 3.1 gives the following result.

Corollary 3.7. *Let $f \in \mathcal{S}_1(0)$. Then we have*

$$\left| f''(z) - \frac{M\beta(1 - |z|^4)}{1 - |\beta|^2 |z|^4} \right| \leq \frac{(1 - |\beta|^2) |z|^2}{1 - |\beta|^2 |z|^4}, \quad z \in \mathbb{D}.$$

For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = F_{e^{i\theta}, 0}$ for some $\theta \in \mathbb{R}$. Here $F_{e^{i\theta}, 0}$ is defined in Theorem 2.11.

For $|\beta| = 1$, by Corollary 3.7, functions in $\mathcal{S}_1(0)$ must satisfy

$$|f''(z) - M\beta| = 0$$

which gives

$$f(z) = \alpha z + \frac{M\beta}{2}z^2.$$

Corollary 3.8. *Let $\gamma : z(t)$, $0 \leq t \leq 1$, be a C^1 -curve in \mathbb{D} with $z(0) = 0$ and $z(1) = z_0$. Then we have*

$$V_1 \subset \overline{\mathbb{D}}(C_1(\lambda, \gamma), R_1(\lambda, \gamma)) = \{w \in \mathbb{C} : |w - C_1(\lambda, \gamma)| \leq R_1(\lambda, \gamma)\},$$

where

$$C_1(\lambda, \gamma) = \alpha + \int_0^1 c_1(z(t), \lambda) z'(t) dt \quad \text{and} \quad R_1(\lambda, \gamma) = \int_0^1 r_1(z(t), \lambda) |z'(t)| dt.$$

Proof. Since for $f \in \mathcal{S}_1$ we have

$$\int_0^1 f''(z(t)) z'(t) dt = f'(z_0) - f'(0) = f'(z_0) - \alpha,$$

it follows from Proposition 3.1 that

$$\begin{aligned} |f'(z_0) - C_1(\lambda, \gamma)| &= \left| f'(z_0) - \alpha - \int_0^1 c_1(z(t), \lambda) z'(t) dt \right| \\ &= \left| \int_0^1 \{f''(z(t)) - c_1(z(t), \lambda)\} z'(t) dt \right| \\ &\leq \int_0^1 r_1(z(t), \lambda) |z'(t)| dt = R_1(\lambda, \gamma). \end{aligned}$$

As $f'(z_0) \in V_1$ was arbitrary, the conclusion follows. \square

Lemma 3.9. *For $\theta \in \mathbb{R}$, $\lambda \in \mathbb{D}$ and $\beta \in \overline{\mathbb{D}}$, the function*

$$G(z) = \int_0^z \frac{e^{i\theta} \zeta^2}{\{1 + (\overline{\lambda} e^{i\theta} + \overline{\beta} \lambda) \zeta + \overline{\beta} e^{i\theta} \zeta^2\}^2} d\zeta, \quad z \in \mathbb{D},$$

has a zero of order three at the origin and no zeros elsewhere in \mathbb{D} . Furthermore, there exists a starlike univalent function G_0 in \mathbb{D} such that $G = 3^{-1} e^{i\theta} G_0^3$ and $G_0(0) = G'_0(0) - 1 = 0$.

Proof. We first prove that

$$(3.10) \quad \operatorname{Re} \left\{ \frac{z G''(z)}{G'(z)} \right\} > -1, \quad z \in \mathbb{D}.$$

If $\beta = 0$, then a simple computation gives (3.10).

If $0 < |\beta| \leq 1$, then it is easy to see that

$$1 + (\overline{\lambda} e^{i\theta} + \overline{\beta} \lambda) z + \overline{\beta} e^{i\theta} z^2 = \left(1 - \frac{z}{z_1}\right) \left(1 - \frac{z}{z_2}\right),$$

where

$$z_1 z_2 = \frac{e^{-i\theta}}{\bar{\beta}} \quad \text{and} \quad z_1 + z_2 = \frac{\bar{\lambda} + e^{-i\theta} \bar{\beta} \lambda}{\bar{\beta}}.$$

We note that $|z_1| |z_2| = 1/|\beta|$. We need to show that none of z_1 and z_2 lie in the punctured unit disk $\mathbb{D} \setminus \{0\}$.

Suppose first that $0 < |\beta| < 1$. Then either both z_1 and z_2 lie outside the unit circle, or else one lies inside while the other lies outside the unit circle. We claim that the later case cannot occur. On the contrary, without loss of generality, we may assume that

$$|z_1| < 1, \quad |z_2| > 1.$$

Also let $z_1 = r e^{i\phi}$, for some $r < 1$ and $\phi \in \mathbb{R}$. Then

$$z_2 = \frac{e^{-i(\theta+\phi)}}{r\bar{\beta}}$$

and so the expression

$$z_1 + z_2 = \frac{\bar{\lambda} + e^{-i\theta} \bar{\beta} \lambda}{\bar{\beta}}$$

simplifies to an equivalent form

$$(3.11) \quad \zeta + \frac{1}{|\beta|\zeta} = - \left(\omega + \frac{1}{|\beta|\omega} \right)$$

with $|\zeta| = 1$ and $|\omega| = |\lambda|$. We may rewrite (3.11) as

$$(\zeta + \omega) \left(1 + \frac{1}{|\beta|\omega\zeta} \right) = 0$$

which is a contradiction, because this equation has no solution when $|\zeta| = 1$ and $|\omega| = |\lambda|$. We conclude that $|z_1| > 1$ and $|z_2| > 1$.

If $|\beta| = 1$, then $|z_1| |z_2| = 1$ so that either $|z_1| = 1$ and $|z_2| = 1$, or $|z_1| < 1$ and $|z_2| > 1$, or $|z_1| > 1$ and $|z_2| < 1$ holds. Again we see that the last two cases cannot occur. Indeed, on the contrary, we may (without loss of generality) assume that

$$|z_1| < 1, \quad |z_2| > 1.$$

Then the expression for $z_1 + z_2$ simplifies to the form

$$(3.12) \quad \zeta + \frac{1}{\zeta} = -\operatorname{Re}(\bar{\lambda} e^{i\psi}),$$

with $|\zeta| = r < 1$ and $\psi \in \mathbb{R}$. Now the set of complex numbers described by the right hand side of (3.12) forms a subset of real numbers lying in the line segment $(-1, 1)$ whereas the set of complex numbers described by the left hand side of (3.12) lies out side of the ellipse

$$\frac{u^2}{(1/4)(r + 1/r)^2} + \frac{v^2}{(1/4)(r - 1/r)^2} = 1.$$

Since the above two sets of complex numbers are disjoint, we arrive at a contradiction. Hence, we conclude that $|z_1| = |z_2| = 1$.

Finally, as $|z_1| \geq 1$ and $|z_2| \geq 1$, a simple calculation shows that

$$\operatorname{Re} \left\{ \frac{zG''(z)}{G'(z)} \right\} = \operatorname{Re} \left(\frac{1+z/z_1}{1-z/z_1} \right) + \operatorname{Re} \left(\frac{1+z/z_2}{1-z/z_2} \right) > 0, \quad z \in \mathbb{D}.$$

Applying Lemma 2.1 to $3e^{-i\theta}G(z)$ with $p = 3$ there exists a $G_0 \in \mathcal{S}^*$ such that $G = 3^{-1}e^{i\theta}G_0^3$. This completes the proof. \square

Proposition 3.13. *Let $z_0 \in \mathbb{D} \setminus \{0\}$. Then for $\theta \in (-\pi, \pi]$ we have $F'_{e^{i\theta}, \lambda}(z_0) \in \partial V_1$. Furthermore if $f'(z_0) = F'_{e^{i\theta}, \lambda}(z_0)$ for some $f \in \mathcal{S}_1$ and $\theta \in (-\pi, \pi]$, then $f = F_{e^{i\theta}, \lambda}$.*

Proof. From (2.5) we easily obtain that

$$F''_{a, \lambda}(z) = \frac{M(\delta(az, \lambda)z + \beta)}{1 + \bar{\beta}\delta(az, \lambda)z} = \frac{M\{(az + \lambda)z + \beta(1 + \bar{\lambda}az)\}}{1 + (\bar{\lambda}a + \bar{\beta}\lambda)z + \bar{\beta}az^2}.$$

Thus we have from (3.5)

$$F''_{a, \lambda}(z) - A_1(z, \lambda) = \frac{M(1 - |\lambda|^2)(1 - |\beta|^2)az^2}{(1 + (\bar{\lambda}a + \bar{\beta}\lambda)z + \bar{\beta}az^2)(1 + \bar{\beta}\lambda z)},$$

$$F''_{a, \lambda}(z) + B_1(z, \lambda) = \frac{-M(1 - |\lambda|^2)(1 - |\beta|^2)z}{(1 + (\bar{\lambda}a + \bar{\beta}\lambda)z + \bar{\beta}az^2)(\bar{\beta}z + \bar{\lambda})}$$

and hence

$$\begin{aligned} & F''_{a, \lambda}(z) - c_1(z, \lambda) \\ &= F''_{a, \lambda}(z) - \frac{A_1(z, \lambda) + |z|^2|\tau_1(z, \lambda)|^2 B_1(z, \lambda)}{1 - |z|^2|\tau_1(z, \lambda)|^2} \\ &= \frac{1}{1 - |z|^2|\tau_1(z, \lambda)|^2} \left\{ (F''_{a, \lambda}(z) - A_1(z, \lambda)) - |z|^2|\tau_1(z, \lambda)|^2 (F''_{a, \lambda}(z) + B_1(z, \lambda)) \right\} \\ &= \frac{M(1 - |\lambda|^2)(1 - |\beta|^2)az^2}{\{(1 - |\beta|^2|z|^4) - (1 - |\beta|^2)|\lambda|^2|z|^2 + 2(1 - |z|^2)\operatorname{Re}(\bar{\beta}\lambda z)\}} \frac{\overline{K(a, z)}}{K(a, z)}, \end{aligned}$$

where

$$K(a, z) = 1 + (\bar{\lambda}a + \bar{\beta}\lambda)z + \bar{\beta}az^2.$$

Substituting $a = e^{i\theta}$, we find that

$$\begin{aligned} & F''_{e^{i\theta}, \lambda}(z) - c_1(z, \lambda) \\ &= \frac{M(1 - |\lambda|^2)(1 - |\beta|^2)e^{i\theta}z^2}{\{(1 - |\beta|^2|z|^4) - (1 - |\beta|^2)|\lambda|^2|z|^2 + 2(1 - |z|^2)\operatorname{Re}(\bar{\beta}\lambda z)\}} \frac{\overline{K(e^{i\theta}, z)}}{K(e^{i\theta}, z)} \\ &= \frac{M(1 - |\lambda|^2)(1 - |\beta|^2)e^{i\theta}z^2}{\{(1 - |\beta|^2|z|^4) - (1 - |\beta|^2)|\lambda|^2|z|^2 + 2(1 - |z|^2)\operatorname{Re}(\bar{\beta}\lambda z)\}} \frac{|K(e^{i\theta}, z)|^2}{(K(e^{i\theta}, z))^2} \\ &= r_1(z, \lambda) \frac{e^{i\theta}z^2}{|z|^2} \frac{|1 + (\bar{\lambda}e^{i\theta} + \bar{\beta}\lambda)z + \bar{\beta}e^{i\theta}z^2|^2}{\{1 + (\bar{\lambda}e^{i\theta} + \bar{\beta}\lambda)z + \bar{\beta}e^{i\theta}z^2\}^2}. \end{aligned}$$

From Lemma 3.9, we may rewrite the last expression as

$$(3.14) \quad F''_{e^{i\theta},\lambda}(z) - c_1(z, \lambda) = r_1(z, \lambda) \frac{G'(z)}{|G'(z)|},$$

where $G(z)$ is defined as in Lemma 3.9. According to Lemma 3.9, the function G_0 defined by $G = 3^{-1}e^{i\theta}G_0^3$ is starlike. As a consequence, for any $z_0 \in \mathbb{D} \setminus \{0\}$, the line segment joining 0 and $G_0(z_0)$ entirely lies in $G_0(\mathbb{D})$. Introduce γ_0 by

$$(3.15) \quad \gamma_0 : z(t) = G_0^{-1}(tG_0(z_0)), \quad 0 \leq t \leq 1.$$

From the representation of G , we obtain

$$G(z(t)) = 3^{-1}e^{i\theta}G_0(z(t))^3 = 3^{-1}e^{i\theta}(tG_0(z_0))^3 = t^3G(z_0)$$

and so,

$$(3.16) \quad G'(z(t))z'(t) = 3t^2G(z_0), \quad t \in [0, 1].$$

Using this and (3.14) we deduce that

$$\begin{aligned} (3.17) \quad F'_{e^{i\theta},\lambda}(z_0) - C_1(\lambda, \gamma_0) &= \int_0^1 \{F''_{e^{i\theta},\lambda}(z(t)) - c_1(z(t), \lambda)\} z'(t) dt \\ &= \int_0^1 r_1(z(t), \lambda) \frac{G'(z(t))z'(t)}{|G'(z(t))z'(t)|} |z'(t)| dt \\ &= \frac{G(z_0)}{|G(z_0)|} \int_0^1 r_1(z(t), \lambda) |z'(t)| dt \\ &= \frac{G(z_0)}{|G(z_0)|} R_1(\lambda, \gamma_0) \end{aligned}$$

which means that $F'_{e^{i\theta},\lambda}(z_0) \in \partial\overline{\mathbb{D}}(C_1(\lambda, \gamma_0), R_1(\lambda, \gamma_0))$. From Corollary 3.8 we also have $F'_{e^{i\theta},\lambda}(z_0) \in V_1 \subset \overline{\mathbb{D}}(C_1(\lambda, \gamma_0), R_1(\lambda, \gamma_0))$ and hence, $F'_{e^{i\theta},\lambda}(z_0) \in \partial V_1$.

Next, we deal with the uniqueness part. Suppose $f'(z_0) = F'_{e^{i\theta},\lambda}(z_0)$ for some $f \in \mathcal{S}_1$ and $\theta \in (-\pi, \pi]$. Define

$$h(t) = \frac{\overline{G(z_0)}}{|G(z_0)|} \{f''(z(t)) - c_1(z(t), \lambda)\} z'(t),$$

where $\gamma_0 : z(t)$, $0 \leq t \leq 1$, as in (3.15). Then $h(t)$ is a continuous function of t on $[0, 1]$ and satisfies the inequality $|h(t)| \leq r_1(z(t), \lambda)|z'(t)|$. Furthermore, from (3.17),

we get

$$\begin{aligned}
\int_0^1 \operatorname{Re} h(t) dt &= \int_0^1 \operatorname{Re} \left\{ \frac{\overline{G(z_0)}}{|G(z_0)|} \{f''(z(t)) - c_1(z(t), \lambda)\} z'(t) \right\} dt \\
&= \operatorname{Re} \left\{ \frac{\overline{G(z_0)}}{|G(z_0)|} \{f'(z_0) - C_1(\lambda, \gamma_0)\} \right\} \\
&= \operatorname{Re} \left\{ \frac{\overline{G(z_0)}}{|G(z_0)|} \{F'_{e^{i\theta}, \lambda}(z_0) - C_1(\lambda, \gamma_0)\} \right\} \\
&= \int_0^1 r_1(z(t), \lambda) |z'(t)| dt
\end{aligned}$$

which shows that $h(t) = r_1(z(t), \lambda) |z'(t)|$ for all $t \in [0, 1]$. From (3.14) and (3.16) this implies $f'' = F''_{e^{i\theta}, \lambda}$ on γ_0 . From the identity theorem for analytic functions we have $f'' = F''_{e^{i\theta}, \lambda}$ in \mathbb{D} and hence by normalization $f = F_{e^{i\theta}, \lambda}$ in \mathbb{D} . \square

4. PREPARATION FOR THE PROOF OF THEOREM 2.12

Proposition 4.1. *For $f \in \mathcal{S}_2$ with $\lambda \in \mathbb{D}$, we have*

$$(4.2) \quad |f''(z) - c_2(z, \lambda)| \leq r_2(z, \lambda), \quad z \in \mathbb{D}$$

where

$$\begin{aligned}
c_2(z, \lambda) &= \frac{2M[(1 - |z|^2)\lambda + (|z|^2 - |\lambda|^2)\bar{z}]}{(1 - |z|^2)(1 + |z|^2 - 2\operatorname{Re}(\lambda z))}, \\
r_2(z, \lambda) &= \frac{2(1 - |\lambda|^2)M|z|}{(1 - |z|^2)(1 + |z|^2 - 2\operatorname{Re}(\lambda z))}.
\end{aligned}$$

For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = H_{e^{i\theta}, \lambda}$ for some $\theta \in \mathbb{R}$.

Proof. Let $f \in \mathcal{S}_2$. Then (1.10) holds with $\omega_f \in \mathcal{B}_0$ and $\omega'_f(0) = \lambda$. It follows from the Schwarz lemma that

$$\left| \frac{\frac{\omega_f(z)}{z} - \lambda}{1 - \bar{\lambda} \frac{\omega_f(z)}{z}} \right| \leq |z|.$$

From (1.9) and (1.10) we see that this equality is same as

$$(4.3) \quad \left| \frac{f''(z) - A_2(z, \lambda)}{f''(z) + B_2(z, \lambda)} \right| \leq |z| |\tau_2(z, \lambda)|,$$

where

$$(4.4) \quad A_2(z, \lambda) = \frac{2M\lambda}{1 - \lambda z}, \quad B_2(z, \lambda) = \frac{2M}{z - \bar{\lambda}} \quad \text{and} \quad \tau_2(z, \lambda) = \frac{z - \bar{\lambda}}{1 - \lambda z}.$$

A computation shows that the inequality (4.3) is equivalent to

$$(4.5) \quad \left| f''(z) - \frac{A_2(z, \lambda) + |z|^2 |\tau_2(z, \lambda)|^2 B_2(z, \lambda)}{1 - |z|^2 |\tau_2(z, \lambda)|^2} \right| \leq \frac{|z| |\tau_2(z, \lambda)| |A_2(z, \lambda) + B_2(z, \lambda)|}{1 - |z|^2 |\tau_2(z, \lambda)|^2}.$$

Also it is easy to obtain that

$$1 - |z|^2 |\tau_2(z, \lambda)|^2 = \frac{(1 - |z|^2)(1 + |z|^2 - 2\operatorname{Re}(\lambda z))}{|1 - \lambda z|^2},$$

$$A_2(z, \lambda) + B_2(z, \lambda) = \frac{2M(1 - |\lambda|^2)}{(1 - \lambda z)(z - \bar{\lambda})},$$

and

$$A_2(z, \lambda) + |z|^2 |\tau_2(z, \lambda)|^2 B_2(z, \lambda) = \frac{2M[(1 - |z|^2)\lambda + (|z|^2 - |\lambda|^2)\bar{z}]}{|1 - \lambda z|^2}.$$

Using these, we obtain that

$$\frac{A_2(z, \lambda) + |z|^2 |\tau_2(z, \lambda)|^2 B_2(z, \lambda)}{1 - |z|^2 |\tau_2(z, \lambda)|^2} = c_2(z, \lambda)$$

and

$$\frac{|z| |\tau_2(z, \lambda)| |A_2(z, \lambda) + B_2(z, \lambda)|}{1 - |z|^2 |\tau_2(z, \lambda)|^2} = r_2(z, \lambda).$$

Now the inequality (4.2) follows from these equalities and (4.5). Final part of the proof, namely, the equality case, follows as in the proof of Proposition 3.1. \square

The case $\lambda = 0$ of Proposition 4.1 gives the following information.

Corollary 4.6. *Let $f \in \mathcal{S}_2(0)$. Then we have*

$$\left| f''(z) - \frac{2M|z|^2 \bar{z}}{1 - |z|^4} \right| \leq \frac{2M|z|}{1 - |z|^4}, \quad z \in \mathbb{D}.$$

For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = H_{e^{i\theta}, 0}$ for some $\theta \in \mathbb{R}$.

In particular, if $f \in \mathcal{S}_2(0)$ then we have

$$(1 - |z|^4) |f''(z)| \leq 2M(1 + |z|^2)|z|, \quad z \in \mathbb{D}$$

and hence

$$\sup_{z \in \mathbb{D}} (1 - |z|^4) |f''(z)| \leq 4M.$$

Corollary 4.7. *Let $\gamma : z(t)$, $0 \leq t \leq 1$, be a C^1 -curve in \mathbb{D} with $z(0) = 0$ and $z(1) = z_0$. Then we have*

$$V_2 \subset \overline{\mathbb{D}}(C_2(\lambda, \gamma), R_2(\lambda, \gamma)) = \{w \in \mathbb{C} : |w - C_2(\lambda, \gamma)| \leq R_2(\lambda, \gamma)\},$$

where

$$C_2(\lambda, \gamma) = \alpha + \int_0^1 c_2(z(t), \lambda) z'(t) dt, \quad R_2(\lambda, \gamma) = \int_0^1 r_2(z(t), \lambda) |z'(t)| dt.$$

Proof. The proof is immediate if one uses Proposition 4.1 and follows the method of proof of Corollary 3.8. So we omit the details. \square

Lemma 4.8. [9] For $\theta \in \mathbb{R}$ and $\lambda \in \mathbb{D}$ the function

$$G(z) = \int_0^z \frac{e^{i\theta}\zeta}{\{1 + (\bar{\lambda}e^{i\theta} - \lambda)\zeta - e^{i\theta}\zeta^2\}^2} d\zeta, \quad z \in \mathbb{D},$$

has a double zero at the origin and no zeros elsewhere in \mathbb{D} . Furthermore there exists a starlike univalent function G_0 in \mathbb{D} such that $G = 2^{-1}e^{i\theta}G_0^2$ and $G_0(0) = G'_0(0) - 1 = 0$.

Proposition 4.9. Let $z_0 \in \mathbb{D} \setminus \{0\}$. Then for $\theta \in (-\pi, \pi]$ we have $H'_{e^{i\theta}, \lambda}(z_0) \in \partial V_2$. Furthermore if $f'(z_0) = H'_{e^{i\theta}, \lambda}(z_0)$ for some $f \in \mathcal{S}_2$ and $\theta \in (-\pi, \pi]$, then $f = H_{e^{i\theta}, \lambda}$.

Proof. From (2.9) we have

$$H''_{a, \lambda}(z) = \frac{2M\delta(az, \lambda)}{1 - \delta(az, \lambda)z} = \frac{2M(az + \lambda)}{1 + (\bar{\lambda}a - \lambda)z - az^2}$$

and using (4.4), we see that

$$H''_{a, \lambda}(z) - A_2(z, \lambda) = \frac{2M(1 - |\lambda|^2)az}{(1 - \lambda z)(1 + (\bar{\lambda}a - \lambda)z - az^2)}$$

and

$$H''_{a, \lambda}(z) + B_2(z, \lambda) = \frac{2M(1 - |\lambda|^2)}{(z - \bar{\lambda})(1 + (\bar{\lambda}a - \lambda)z - az^2)}.$$

Using these, it follows that

$$\begin{aligned} H''_{a, \lambda}(z) - c_2(z, \lambda) &= H''_{a, \lambda}(z) - \frac{A_2(z, \lambda) + |z|^2|\tau_2(z, \lambda)|^2 B_2(z, \lambda)}{1 - |z|^2|\tau_2(z, \lambda)|^2} \\ &= \frac{1}{1 - |z|^2|\tau_2(z, \lambda)|^2} \{H''_{a, \lambda}(z) - A_2(z, \lambda) - |z|^2|\tau_2(z, \lambda)|^2 (H''_{a, \lambda}(z) + B_2(z, \lambda))\} \\ &= \frac{2M(1 - |\lambda|^2)az\{1 + (\bar{\lambda}a - \lambda)z - az^2\}}{(1 - |z|^2)\{1 + |z|^2 - 2\operatorname{Re}(\lambda z)\}\{1 + (\bar{\lambda}a - \lambda)z - az^2\}}. \end{aligned}$$

Substituting $a = e^{i\theta}$, a computation gives

$$\begin{aligned} H''_{e^{i\theta}, \lambda}(z) - c_2(z, \lambda) &= \frac{2M(1 - |\lambda|^2)e^{i\theta}z\{1 + (\bar{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2\}}{(1 - |z|^2)\{1 + |z|^2 - 2\operatorname{Re}(\lambda z)\}\{1 + (\bar{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2\}} \\ &= \frac{2M(1 - |\lambda|^2)e^{i\theta}z|1 + (\bar{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2|^2}{(1 - |z|^2)\{1 + |z|^2 - 2\operatorname{Re}(\lambda z)\}\{1 + (\bar{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2\}^2} \\ &= r_2(z, \lambda) \frac{|1 + (\bar{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2|^2}{|z|} \frac{e^{i\theta}z}{\{1 + (\bar{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2\}^2}. \end{aligned}$$

Using Lemma 4.8, we may rewrite the last equality as

$$(4.10) \quad H''_{e^{i\theta}, \lambda}(z) - c_2(z, \lambda) = r_2(z, \lambda) \frac{G'(z)}{|G'(z)|}$$

where $G(z)$ is defined as in Lemma 4.8. Since G_0 defined by $G = 2^{-1}e^{i\theta}G_0^2$ is a normalized starlike function, for any $z_0 \in \mathbb{D} \setminus \{0\}$, the line segment joining 0 and $G_0(z_0)$ entirely lies in $G_0(\mathbb{D})$. As before, define γ_0 by

$$\gamma_0 : z(t) = G_0^{-1}(tG_0(z_0)), \quad 0 \leq t \leq 1.$$

We observe that $G(z(t)) = 2^{-1}e^{i\theta}G_0(z(t))^2 = 2^{-1}e^{i\theta}(tG_0(z_0))^2 = t^2G(z_0)$ and so, we get

$$(4.11) \quad G'(z(t))z'(t) = 2tG(z_0), \quad t \in [0, 1].$$

From this, (4.10) and proceeding exactly as in the proof of Proposition 3.13, we end up with

$$(4.12) \quad H'_{e^{i\theta}, \lambda}(z_0) - C_2(\lambda, \gamma_0) = \frac{G(z_0)}{|G(z_0)|} R_2(\lambda, \gamma_0)$$

which gives $H'_{e^{i\theta}, \lambda}(z_0) \in \partial \overline{\mathbb{D}}(C_2(\lambda, \gamma_0), R_2(\lambda, \gamma_0))$. From Corollary 4.7, we also have $H'_{e^{i\theta}, \lambda}(z_0) \in V_2 \subset \overline{\mathbb{D}}(C_2(\lambda, \gamma_0), R_2(\lambda, \gamma_0))$. Hence, $H'_{e^{i\theta}, \lambda}(z_0) \in \partial V_2$.

Uniqueness part follows similarly. Indeed, suppose $f'(z_0) = H'_{e^{i\theta}, \lambda}(z_0)$ for some $f \in \mathcal{S}_2$ and $\theta \in (-\pi, \pi]$ and introduce,

$$h(t) = \frac{\overline{G(z_0)}}{|G(z_0)|} \{f''(z(t)) - c_2(z(t), \lambda)\} z'(t).$$

Then $h(t)$ is continuous function of $t \in [0, 1]$ and satisfies $|h(t)| \leq r_2(z(t), \lambda)|z'(t)|$. Furthermore, from (4.12), we easily see that

$$\int_0^1 \operatorname{Re} h(t) dt = \int_0^1 r_2(z(t), \lambda)|z'(t)| dt.$$

Thus, $h(t) = r_2(z(t), \lambda)|z'(t)|$ for all $t \in [0, 1]$. From (4.10) and (4.11) this implies that $f'' = H''_{e^{i\theta}, \lambda}$ on γ_0 . From the identity theorem for analytic functions we deduce that $f'' = H''_{e^{i\theta}, \lambda}$ in \mathbb{D} and hence by normalization $f = H_{e^{i\theta}, \lambda}$ in \mathbb{D} . \square

5. PROOFS OF THEOREMS 2.11 AND 2.12

5.1. Proof of Theorem 2.11 . We prove that the closed curve $(-\pi, \pi] \ni \theta \mapsto F'_{e^{i\theta}, \lambda}(z_0)$ is simple. Suppose that $F'_{e^{i\theta_1}, \lambda}(z_0) = F'_{e^{i\theta_2}, \lambda}(z_0)$ for some $\theta_1, \theta_2 \in (-\pi, \pi]$ with $\theta_1 \neq \theta_2$. Then, from Proposition 3.13, we conclude that $F_{e^{i\theta_1}, \lambda} = F_{e^{i\theta_2}, \lambda}$.

From (2.6) and (3.5) we have

$$\tau_1 \left(\frac{\omega_{F_{e^{i\theta}, \lambda}}}{z}, \lambda \right) = \frac{(\bar{\beta} + \bar{\lambda}^2)e^{i\theta}z + (\bar{\beta}\lambda + \bar{\lambda})}{(\bar{\lambda} + \bar{\beta}\lambda)e^{i\theta}z + (\bar{\beta}\lambda^2 + 1)}.$$

Since $F_{e^{i\theta_1}, \lambda} = F_{e^{i\theta_2}, \lambda}$, we have the following relation

$$\tau_1 \left(\frac{\omega_{F_{e^{i\theta_1}, \lambda}}}{z}, \lambda \right) = \tau_1 \left(\frac{\omega_{F_{e^{i\theta_2}, \lambda}}}{z}, \lambda \right).$$

That is

$$\frac{(\bar{\beta} + \bar{\lambda}^2)e^{i\theta_1}z + (\bar{\beta}\lambda + \bar{\lambda})}{(\bar{\lambda} + \bar{\beta}\lambda)e^{i\theta_1}z + (\bar{\beta}\lambda^2 + 1)} = \frac{(\bar{\beta} + \bar{\lambda}^2)e^{i\theta_2}z + (\bar{\beta}\lambda + \bar{\lambda})}{(\bar{\lambda} + \bar{\beta}\lambda)e^{i\theta_2}z + (\bar{\beta}\lambda^2 + 1)}.$$

By a simplification, the last expression implies

$$e^{i\theta_1}z = e^{i\theta_2}z$$

which is a contradiction for the choice of θ_1 and θ_2 . Thus the curve is simple.

Since V_1 is a compact convex subset of \mathbb{C} and has nonempty interior, the boundary ∂V_1 contains the curve $(-\pi, \pi] \ni \theta \mapsto F'_{e^{i\theta}, \lambda}(z_0)$. Note that a simple closed curve cannot contain any simple closed curve other than itself. Thus ∂V_1 is given by $(-\pi, \pi] \ni \theta \mapsto F'_{e^{i\theta}, \lambda}(z_0)$.

5.2. Proof of Theorem 2.12 . We prove that the closed curve $(-\pi, \pi] \ni \theta \mapsto H'_{e^{i\theta}, \lambda}(z_0)$ is simple. Suppose that $H'_{e^{i\theta_1}, \lambda}(z_0) = H'_{e^{i\theta_2}, \lambda}(z_0)$ for some $\theta_1, \theta_2 \in (-\pi, \pi]$ with $\theta_1 \neq \theta_2$. Then from Proposition 4.9 we have $H_{e^{i\theta_1}, \lambda} = H_{e^{i\theta_2}, \lambda}$. From (2.10) and (4.4) this shows a contradiction

$$e^{i\theta_1}z = \tau_2 \left(\frac{\omega_{H_{e^{i\theta_1}, \lambda}}}{z}, \lambda \right) = \tau_2 \left(\frac{\omega_{H_{e^{i\theta_2}, \lambda}}}{z}, \lambda \right) = e^{i\theta_2}z.$$

Thus the curve is simple.

Again, since V_2 is a compact convex subset of \mathbb{C} and has nonempty interior, the boundary ∂V_2 contains the curve $(-\pi, \pi] \ni \theta \mapsto H'_{e^{i\theta}, \lambda}(z_0)$. The same reasoning as in the proof of Theorem 2.11 shows that ∂V_2 is given by $(-\pi, \pi] \ni \theta \mapsto H'_{e^{i\theta}, \lambda}(z_0)$.

6. GEOMETRIC VIEW OF THEOREMS 2.11 AND 2.12

Using Mathematica 4.1, we describe the boundary of the sets $V_j(z_0, \lambda)$ for $j = 1, 2$. Here we give the Mathematica program which is used to plot the boundary of the sets $V_j(z_0, \lambda)$ for $j = 1, 2$. We refer [10] for Mathematica program. The short notations in this program are of the form: “z0 for z_0 ”, “a for α ”, “lam for λ ”, “m for M ” and “b for β ”.

```
Remove["Global`*"];
(* The values ‘‘z0, a, lam, m, b’’ are for FIGURE 1 *)
z0 = 0.00882581 - 0.514124I
a = -230.939 + 799.526I
lam = 0.427174 + 0.0755107I
m = 509.317
b = 0.94485 + 0.0416585I

Q1[b_, m_, lam_, the_] :=
m((Exp[I*the]z + lam)z + b(1 + Conjugate[lam]Exp[I*the]z))/
((1 + (Conjugate[lam]*Exp[I*the] + Conjugate[b]*lam)*z) +
Conjugate[b]*Exp[I*the]*z*z);
```

```

myf1[a_, b_, m_, lam_, the_, z0_] :=
a + NIntegrate[Q1[b, m, lam, the], {z, 0, z0}];

image1 = ParametricPlot[{Re[myf1[a, b, m, lam, the, z0]],
    Im[myf1[a, b, m, lam, the, z0]]}, {the, -Pi, Pi},
    AspectRatio -> Automatic];

Clear[a, b, m, lam, the, z0, myf1];

z0 = 0.00882581 - 0.514124I
a = -230.939 + 799.526I
lam = 0.839567
m = 0.254877

Q2[m_, lam_, the_] :=
2*m*(Exp[I*the]*z + lam)/(1 + lam*(Exp[I*the] - 1)*z
- Exp[I*the]*z*z);

myf2[a_, m_, lam_, the_, z0_] :=
a + NIntegrate[Q2[m, lam, the], {z, 0, z0}];

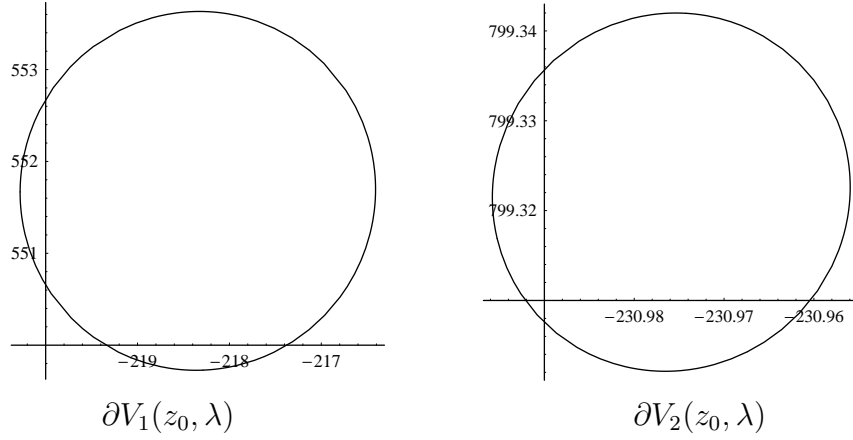
image2 = ParametricPlot[{Re[myf2[a, m, lam, the, z0]],
    Im[myf2[a, m, lam, the, z0]]}, {the, -Pi, Pi},
    AspectRatio -> Automatic];

image=Show[GraphicsArray[{image1,image2},GraphicsSpacing --->0.5]]

Clear[a, m, lam, the, z0, myf2];

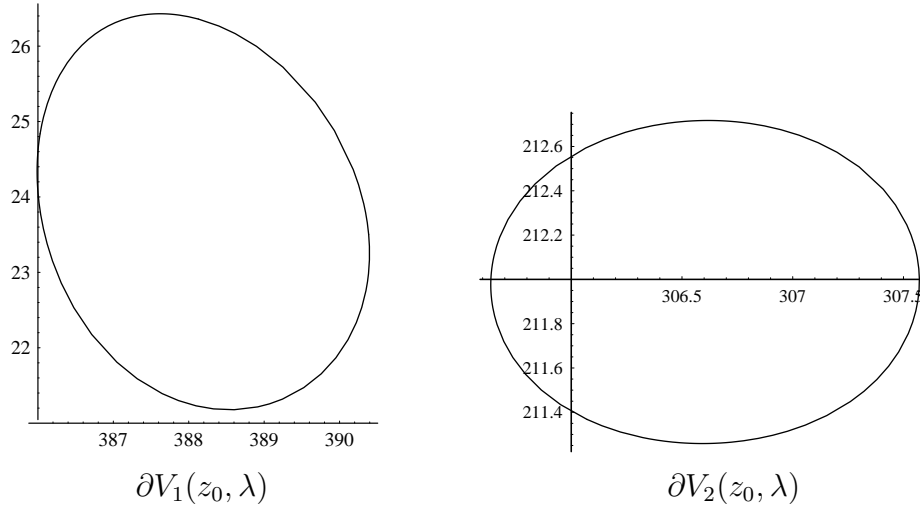
```

The following pictures give the geometric view of the boundary of the sets $V_j(z_0, \lambda)$ for each $j = 1, 2$. In each of the following figures the left hand side figure describes the boundary of the set $V_1(z_0, \lambda)$ for each fixed value of $z_0 \in \mathbb{D} \setminus \{0\}$, $\lambda \in \mathbb{D}$, $\alpha, \beta \in \mathbb{C}$ and $M \in \mathbb{R}$ with $0 < M \leq |\alpha|$ and $|\beta| \leq 1$. These values are given in the first column of each the figure. Similarly the right hand side of each of the following figures describes the boundary of the set $V_2(z_0, \lambda)$ for each fixed value of $z_0 \in \mathbb{D} \setminus \{0\}$, $\lambda \in [0, 1)$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $M \in \mathbb{R}$ such that $0 < M \leq 1/\log 4$ and these values are given in the second column of each of the figures. Note that according to Proposition 2.2 and Proposition 2.7 the regions bounded by the curves $\partial V_j(z_0, \lambda)$ for $j = 1, 2$ are compact and convex.

FIGURE 1. Region of variability for $f'(z_0)$

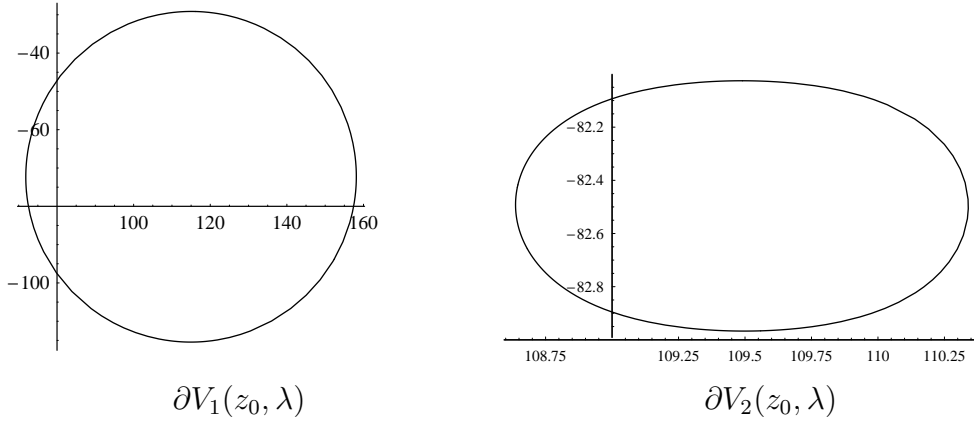
$$\begin{aligned}
 z_0 &= 0.00882581 - 0.514124i \\
 \alpha &= -230.939 + 799.526i \\
 \lambda &= 0.427174 + 0.0755107i \\
 M &= 509.317 \\
 \beta &= 0.94485 + 0.0416585i
 \end{aligned}$$

$$\begin{aligned}
 z_0 &= 0.00882581 - 0.514124i \\
 \alpha &= -230.939 + 799.526i \\
 \lambda &= 0.839567 \\
 M &= 0.254877
 \end{aligned}$$

FIGURE 2. Region of variability for $f'(z_0)$

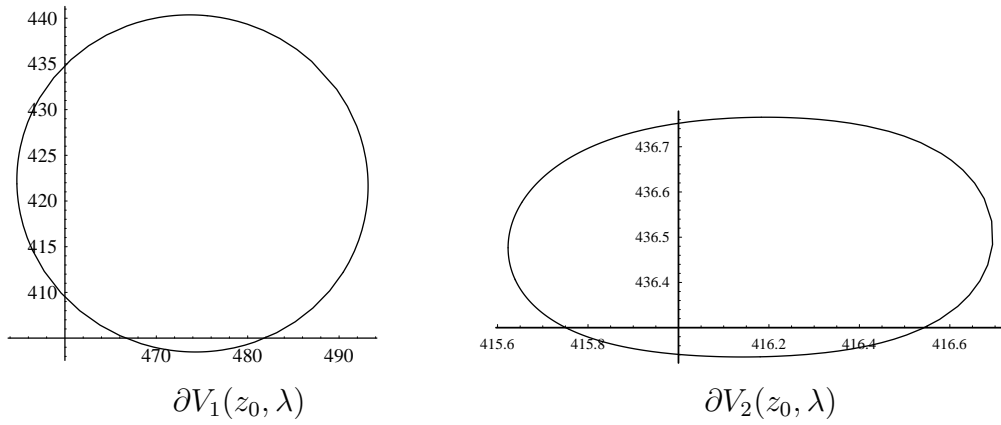
$$\begin{aligned}
 z_0 &= -0.439619 - 0.843107i \\
 \alpha &= 306.095 + 212.047i \\
 \lambda &= -0.847689 - 0.07592i \\
 M &= 206.329 \\
 \beta &= 0.67079 + 0.843107i
 \end{aligned}$$

$$\begin{aligned}
 z_0 &= -0.439619 - 0.843107i \\
 \alpha &= 306.095 + 212.047i \\
 \lambda &= 0.0802624 \\
 M &= 0.673609
 \end{aligned}$$

FIGURE 3. Region of variability for $f'(z_0)$

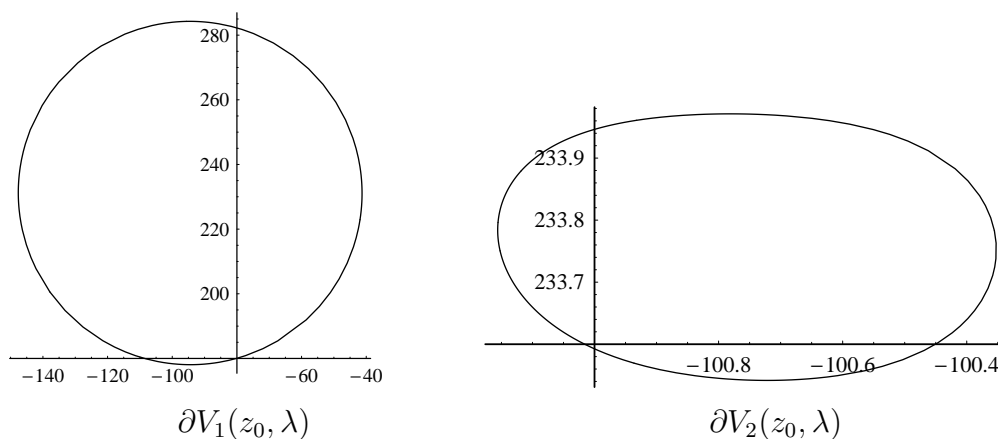
$$\begin{aligned} z_0 &= -0.971007 + 0.211382i \\ \alpha &= 108.958 - 82.5096i \\ \lambda &= 0.0327389 - 0.0219389i \\ M &= 132.988 \\ \beta &= -0.0264629 - 0.114565i \end{aligned}$$

$$\begin{aligned} z_0 &= -0.971007 + 0.211382i \\ \alpha &= 108.958 - 82.5096i \\ \lambda &= 0.148939 \\ M &= 0.390188 \end{aligned}$$

FIGURE 4. Region of variability for $f'(z_0)$

$$\begin{aligned} z_0 &= -0.844358 - 0.529996i \\ \alpha &= 416.349 + 436.752i \\ \lambda &= -0.0872118 + 0.664418i \\ M &= 97.2626 \\ \beta &= -0.549327 + 0.592394i \end{aligned}$$

$$\begin{aligned} z_0 &= -0.844358 - 0.529996i \\ \alpha &= 416.349 + 436.752i \\ \lambda &= 0.7262 \\ M &= 0.620559 \end{aligned}$$

FIGURE 5. Region of variability for $f'(z_0)$

$$\begin{aligned}
 z_0 &= -0.605185 + 0.789592i \\
 \alpha &= -100.796 + 233.556i \\
 \lambda &= 0.0523661 + 0.167249i \\
 M &= 164.079 \\
 \beta &= 0.00810121 - 0.00819085i
 \end{aligned}$$

$$\begin{aligned}
 z_0 &= -0.605185 + 0.789592i \\
 \alpha &= -100.796 + 233.556i \\
 \lambda &= 0.63945 \\
 M &= 0.354197
 \end{aligned}$$

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